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Rigorous Results for General Ising Ferromagnets

Charles M. Newman[1,2]

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Several new results are given concerning the Lee–Yang theorem, the GHS inequality, and spin-$\frac{1}{2}$ approximations for general Ising ferromagnets, and the extension of these results to vector spin models is discussed.

KEY WORDS: Spin systems; general Ising ferromagnets; Lee–Yang theorem; vector spin models.

1. INTRODUCTION

Consider a system of classical single-component spins S_i, labeled by points i in some finite lattice Λ, for which the thermal average at inverse temperature $\beta = 1/kT$ of any function $F(\{S_i\})$ of the spins is given by

$$\langle F \rangle = Z^{-1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(\{s_i\}) \exp(-\beta \mathscr{H}) \prod_i d\rho(s_i) \tag{1}$$

where the Hamiltonian $\mathscr{H}$ has the form

$$\mathscr{H} = -\sum_{i,j} J_{ij} S_i S_j - \sum_i h_i S_i \tag{2}$$

and the partition function Z is given by

$$Z = Z(\{h_i\}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[-\beta \mathscr{H}(\{s_i\})] \prod_i d\rho(s_i) \tag{3}$$

Here ρ is some even positive measure on the real line [assumed to satisfy $\int \exp(bs^2)\, d\rho(s) < \infty$ for some $b > 0$ so that Z will be finite at least for small

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[1] Departments of Physics and Mathematics, Technion–Israel Institute of Technology, Haifa, Israel.

[2] NATO postdoctoral fellow on leave from the Department of Mathematics, Indiana University, Bloomington, Indiana.

β] which is independent of β and is determined by the "internal" properties of the spins.

Such a system constitutes a (finite) general Ising model with pair interactions J_{ij} in an external magnetic field h_i. In spin-$K/2$ models, where each spin can only have the values $K, K-2, \dots, -K$,

$$\rho(s) = \delta(s-K) + \delta(s-K+2) + \cdots + \delta(s+K)$$

while in "classical" continuous spin Ising models,

$$d\rho/ds = \begin{cases} 1, & |s| \leqslant 1 \\ 0, & |s| > 1 \end{cases} \tag{4}$$

General Ising models have become the object of considerable study during recent years in constructive quantum field theory, where they represent the "lattice approximations" to various Euclidean field models[1]; there $d\rho/ds$ typically has the form $\exp[-\lambda V(s)]$, where V is a polynomial or some other analytic function such as $\cosh s$. This paper will only deal with ferromagnetic models, i.e., those in which $J_{ij} \geqslant 0$ for $i \neq j$, but the sign of the "self-interaction" couplings J_{ii} will be unrestricted. This freedom for the J_{ii}, although irrelevant for spin-$\frac{1}{2}$ models (where $S_i^2 = 1$ so that the self-coupling terms only contribute an overall constant to $\mathscr{H}$), is natural for certain spin-1 models of liquid helium mixtures and ternary alloys[2] as well as for quantum field models.[1]

Essentially all rigorous results for Ising models were first proved in the spin-$\frac{1}{2}$ case and our main concern in this paper is to determine the classes of ρ's to which various of these results extend. We will consider in Section 2 the Lee–Yang "circle" theorem[3] and in Section 3 the GHS inequality,[4] but will not discuss at all those results (such as the GKS and FKG correlation inequalities) which are valid for all ρ's. The characterizations given in the theorems of Sections 2 and 3 are complete and explicit; the proofs of these theorems are only sketched to avoid lengthy discussions of technical details. We will then consider in Section 4 a class of ρ's to which essentially all spin-$\frac{1}{2}$ results apply. The measure, $\exp(-as^4 - bs^2)\,ds$ with $a > 0$, was shown by Simon and Griffiths[5] to belong to this class and each of the theorems in this paper can be considered as an extension and/or clarification of their result. It should be pointed out that our results come out particularly neatly precisely because we do not restrict the sign of the J_{ii}; for a review of the situation under the requirement $J_{ii} \geqslant 0$ see Ref. 6. In Section 5 we discuss the extension of the theorems of this paper to vector spin models.

2. LEE–YANG PROPERTY

Lee and Yang[3] proved that for a spin-$\frac{1}{2}$ Ising model with pair ferromagnetic interactions, the zeros in the complex h plane of the partition function $Z(\{h\lambda_i\})$ (for any fixed nonnegative λ_i) are all pure imaginary (in a lattice gas model, the imaginary axis for h is transformed into the unit circle for the activity variable); we will say that a partition function $Z(\{h_i\})$ has the Lee–Yang property if it also satisfies the above conclusions of the spin-$\frac{1}{2}$ "circle" theorem. The following theorem gives necessary and sufficient conditions for the Lee–Yang property.

Theorem 1. $Z(\{h_i\})$ will have the Lee–Yang property for any choice of J_{ij} (with $J_{ij} \geqslant 0$ for $i \neq j$ and J_{ii} arbitrary) if and only if either

$$\rho = C[\delta(s - s_0) + \delta(s + s_0)] \tag{5}$$

for some $C > 0$ and $s_0 \geqslant 0$, or else

$$d\rho/ds = Cs^{2m} \exp(-as^4 - bs^2) \prod_j (1 + s^2/\alpha_j{}^2) \exp(-s^2/\alpha_j{}^2) \tag{6}$$

for some $C > 0$, $m = 0, 1, 2,\ldots$, $a \geqslant 0$, b arbitrary, and real α_j with $\sum_j (1/\alpha_j)^4 < \infty$.

Proof. It follows from the results of Ref. 7 that it is both necessary and sufficient that $\int \exp(hs - cs^2)\, d\rho(s)$ have only pure imaginary zeros in h for any $c \geqslant 0$. The class of ρ's satisfying this condition is shown in Ref. 8 to consist exactly of those given in (5) and (6); for convenience, we now give a short sketch of the proof. The key ingredient is to write ρ as the limit of measures ρ_c (as $c \to \infty$), where

$$d\rho_c/ds = (c/\pi)^{1/2} \int \exp[-c(s - t)^2]\, d\rho(t) = F_c(s) \exp(-cs^2) \tag{7}$$

with F_c proportional to $\int \exp(2cst - ct^2)\, d\rho(t)$ and thus having only pure imaginary zeros in s, so that, by Ref. 9, Prop. 2,

$$d\rho_c/ds = K_c \exp(-d_c s^2) \prod_j [1 + s^2/\alpha_j(c)^2] \tag{8}$$

Since such a ρ_c is known to yield the Lee–Yang property (e.g., by Ref. 7, Prop. 2.4), the remainder of the proof consists in showing that the limits of such ρ_c's are exactly the measures of (5) and (6). A complete proof of this last fact is given in Ref. 8; we only note here that

$$(1 + x^2/n^{1/2})^n \exp(-n^{1/2}x^2) \to \exp(-x^4/2).$$

Remark. One immediate consequence of this theorem, which is relevant in the context of quantum field theory, is that for $d\rho/ds = \exp[-\lambda V(s) - cs^2]$, the Lee–Yang property is valid for all λ, $c > 0$ if and only if $V(s) = as^4 + bs^2$. This disproves, for example, the conjecture of Ref. 6 that $V(s) = \cosh s + bs^2$ also gives such a Lee–Yang property.

3. GHS INEQUALITY

The GHS inequality[4] states that for any choice of i, j, k,

$$(\partial^3/\partial h_i\, \partial h_j\, \partial h_k) \ln Z(\{h_l\}) \leqslant 0 \qquad \text{when } h_l \geqslant 0 \quad \text{for all } l;$$

it was first proved for spin-$\frac{1}{2}$ models and implies that the magnetization is a concave function of the (positive) external field. The next theorem, which gives necessary and sufficient conditions for the GHS inequality, is taken from Ref. 10.

Theorem 2. The GHS inequality is valid for any choice of J_{ij} (with $J_{ij} \geqslant 0$ for $i \neq j$ and J_{ii} arbitrary) if and only if either ρ is as in (5) or else

$$d\rho/ds = \begin{cases} C \exp[-\int_0^s g(t)\, dt], & |s| < A \\ 0, & |s| \geqslant A \end{cases} \tag{9}$$

with $C > 0$, $0 < A \leqslant +\infty$, and g an odd function [with $g(0) = 0$] which is convex on $[0, A)$.

Proof. The proof that the GHS inequality is valid for ρ as in (9) was given in Ref. 11 for continuous g and extended in Ref. 10 to the general case. To see that ρ must either be as in (5) or (9), it suffices to show that these are the only measures such that

$$(d^3/dh^3) \ln \int \exp(hs - cs^2)\, d\rho(s) \leqslant 0 \qquad \text{when} \quad h \geqslant 0$$

for any $c \geqslant 0$; we proceed to sketch a proof which is given in complete detail in Ref. 10. As in Theorem 1, $\rho = \lim_{c\to\infty} \rho_c$ with ρ_c given by (7), but this time we note that $(d^3/ds^3) \ln F_c(s) \leqslant 0$ for $s \geqslant 0$, so that

$$d\rho_c/ds = K_c \exp\left[-\int_0^s g_c(t)\, dt\right]$$

with g_c odd and convex on $[0, \infty)$; the remainder of the proof consists in showing that the limits of such ρ_c's are exactly the measures of (5) and (9).

Remark. The Ising models of Theorem 2 not only satisfy the GHS inequality, but also the other correlation inequalities of Ref. 12; this can be seen by combining the results of Refs. 11 and 13. There also exists a class of Ising models which satisfy a "reverse" GHS inequality; we do not include the analog of Theorem 2 for these models[14] since they seem to be of little physical interest.

4. SPIN-$\frac{1}{2}$ APPROXIMATIONS

There exist spin-$\frac{1}{2}$ results, such as the Gaussian correlation inequalities of Ref. 15 and the Ursell function inequality of Refs. 16–18, which cannot be extended to general Ising models through the use of the above theorems. To extend these results from the spin-$\frac{1}{2}$ case, we now consider a class of ρ's to which essentially all known spin-$\frac{1}{2}$ results extend; these are measures which can be constructed out of (or at least approximated by) spin-$\frac{1}{2}$ models in the spirit of Griffiths' "analog" spin-$\frac{1}{2}$ systems.[5,19] We call such a ρ ferromagnetic (in Ref. 15 it is called mean zero ferromagnetic) and define it formally as a measure such that there exists a sequence (indexed by n) of (finite) "analog" spin-$\frac{1}{2}$ Ising models [with lattice index j in $\Lambda(n)$] with pair ferromagnetic interactions so that for some choice of $\lambda_j(n) \geqslant 0$,

$$Z_n(\{h\lambda_j(n)\})/Z_n(\{0\}) \to \int \exp(hs)\, d\rho(s) \Big/ \int d\rho(s) \tag{10}$$

as $n \to \infty$ (with the convergence uniform on bounded subsets of the complex h plane).

Griffiths proved that the spin-$K/2$ measures and the "classical" continuous spin measure of (4) are ferromagnetic,[19] so that the results of ferromagnetic spin-$\frac{1}{2}$ models extend to such models (at least when $J_{ij} \geqslant 0$, *including* $i = j$). In order for spin-$\frac{1}{2}$ results to be valid with arbitrarily negative J_{ii}, we should choose ρ's such that $[\exp(-cs^2)]\rho(s)$ is ferromagnetic for any $c \geqslant 0$; we will denote by $\mathscr{F}_s$ the class of such ρ's. The first nontrivial examples of measures in $\mathscr{F}_s$, discovered by Simon and Griffiths,[5] are given by $d\rho/ds = \exp(-as^4 - bs^2)$ with $a > 0$. The next theorem states various properties of $\mathscr{F}_s$; its main interest, however, lies in the fact that it yields an amazingly simple proof of the Simon–Griffiths result and, as we explain below, of the related Dunlop–Newman result for vector spin models.[20]

Theorem 3. The measure of (5) is in $\mathscr{F}_s$; in order for any other ρ to belong to $\mathscr{F}_s$, it is necessary that $d\rho = f(s)\, ds$ with $f(s) \equiv d\rho/ds$ satisfying the conditions of *both* (6) and (9). If $\rho(s)$ and $f(s)\, ds$ are in $\mathscr{F}_s$ and μ is ferromagnetic, then the following are also in $\mathscr{F}_s$:

$$\left[\int \exp(Jst)\, d\mu(t)\right] \cdot \rho(s) \qquad \text{for} \quad J \geqslant 0 \tag{11}$$

(assuming that this defines a finite measure) and

$$f(s)\rho(s) \tag{12}$$

In addition, the following measures belong to $\mathscr{F}_s$:

$$K \exp(-cs^2/2)\, ds \qquad \text{for} \quad K, c > 0 \tag{13}$$

$$K \exp(-as^4 - bs^2)\, ds \qquad \text{for} \quad K, a > 0 \tag{14}$$

Finally, we note that any ν in $\mathscr{F}_s$ can be obtained as the limit of measures as in (11) with μ in $\mathscr{F}_s$ and ρ as in (13).

Proof. The first part of the theorem follows easily from the definition of $\mathscr{F}_s$ together with Theorems 1 and 2. We next note that a measure ν with

$$\int \exp(hs)\, d\nu(s) = \iint \exp(h\lambda_1 s_1 + h\lambda_2 s_2 + Js_1 s_2)\, d\mu_1(s_1)\, d\mu_2(s_2) \quad (15)$$

will be ferromagnetic providing $J, \lambda_1, \lambda_2 \geqslant 0$, and the μ_i are ferromagnetic; this is so because the two analog spin-$\frac{1}{2}$ systems for the μ_i can be combined in an obvious way to give an analog system for ν. By choosing $\lambda_1 = 1$, $\lambda_2 = 0$, $\mu_1 = \rho$, and $\mu_2 = \mu$, we obtain (11). If μ in (11) is in $\mathscr{F}_s$, then we also clearly have that $(d\mu_c/ds)\cdot\rho(s)$ is in $\mathscr{F}_s$ with μ_c defined as in (7); taking $d\mu = f(s)\, ds$ and letting $c \to \infty$ then gives (12). To see that (13) belongs to $\mathscr{F}_s$, we note that

$$\exp(h^2/2c) = \lim_{n\to\infty} [\cosh(h/cn^{1/2})]^n \quad (16)$$

so that in line with (10) we choose the analog systems to consist of n uncoupled spins in a constant external field, $h_i = h/cn^{1/2}$. To obtain (14) we apply formula (11) m times with $\mu(t) = [\delta(t-1) + \delta(t+1)]/2$ and starting with ρ as in (13) to see that

$$K[\cosh(Js)]^m \exp(-cs^2/2)\, ds$$

is in $\mathscr{F}_s$; letting $J = (12a/m)^{1/4}$, $c = 2[(12am)^{1/2} + b]$, and taking the limit as $m \to \infty$ [by writing out the Taylor series for $\ln\cosh(Js)$] yields (14) as desired. The final statement of the theorem follows directly by writing $\nu = \lim_{c\to\infty} \nu_c$ with ν_c defined as in (7).

Remark. An amusing example of a measure in $\mathscr{F}_s$ different than (but related to) (14) can be obtained by using the methods of Ref. 10. Namely, if $\Omega(s)$ is the (positive) ground-state wave function of a quartic anharmonic oscillator Hamiltonian, $-(d^2/ds^2) + As^4 + Bs^2$ (with $A > 0$), then $\Omega(s)\, ds \in \mathscr{F}_s$. The resulting infinite product representation of Ω, based on (6), seems potentially useful for numerical analysis purposes. A simple extension of Theorem 1 shows that a similar representation is also valid for the first excited state.

5. VECTOR SPIN MODELS

In this section, we briefly discuss the situation as regards vector spin models where the S_i are replaced by D-component spins $\mathbf{S}_i = (S_i^1, \ldots, S_i^D)$. For simplicity we suppose that the measure $\rho(s_i)$ of (3) is replaced by a spherically symmetric $\rho(\mathbf{s}_i)$, and that the Hamiltonian of (2) is replaced by

$$-\sum_{i,j} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j - \sum_i h_i S_i^1$$

A Lee–Yang theorem is presently only known to be valid for $D \leqslant 3$ and only for $d\rho(\mathbf{s}) = C\,\delta(|\mathbf{s}| - s_0)$ (classical rotator models) or measures, such as $d\rho(\mathbf{s}) = \exp(-a|\mathbf{s}|^4 - b|\mathbf{s}|^2)\,d\mathbf{s}$, which can be constructed from "analog" classical rotator models.[20] The results of Ref. 7 have unfortunately not been extended to $D > 1$ to provide sufficient conditions for the validity of the Lee–Yang property in terms of the zeros of $\int \exp(hs^1)\,d\rho(\mathbf{s})$; thus the sufficiency part of Theorem 1 has at present no equivalent result for $D > 1$. On the other hand, in the context of Theorem 1, it is clearly necessary that $\phi_c(h) \equiv \int \exp(hs^1 - c|\mathbf{s}|^2)\,d\rho(\mathbf{s})$ have only pure imaginary zeros in h for every $c \geqslant 0$, so that since, by spherical symmetry, $\int \exp(\mathbf{s}\cdot\mathbf{t} - c|\mathbf{t}|^2)\,d\rho(\mathbf{s}) = \phi_c(|\mathbf{t}|)$, we may apply the methods of Theorem 1 to conclude that ρ must either be a classical rotator or else $d\rho/d\mathbf{s}$ must have the form of (6) with s replaced by $|\mathbf{s}|$.

The status of Theorem 2 is that at present no version of the GHS inequalities are known with $D > 1$. For a discussion of the status of other correlation inequalities, see Refs. 20–23.

The notion of ferromagnetic measures is extended to $D > 1$ in a straightforward manner by considering "analog" classical rotator systems. Theorem 3 then carries over *in toto* with certain obvious changes, such as replacing s by $|\mathbf{s}|$; the proof is essentially identical except that

$$\cosh h \equiv \int [\exp(hs)][\delta(s-1) + \delta(s+1)]\,ds/2$$

is replaced by

$$\psi(|\mathbf{h}|) \equiv \int [\exp(\mathbf{h}\cdot\mathbf{s})]\,\delta(|\mathbf{s}| - 1)\,d\mathbf{s}.$$

We thus obtain a significant simplification of the Dunlop–Newman proof that $\exp(-a|\mathbf{s}|^4 - b|\mathbf{s}|^2)\,d\mathbf{s}$ has a classical rotator approximation.

REFERENCES

1. F. Guerra, L. Rosen, and B. Simon, The $P(\varphi)_2$ Euclidean Quantum Field Theory as Classical Statistical Mechanics, *Ann. Math.* **101**:111–259 (1975).
2. M. Blume, V. J. Emery, and R. B. Griffiths, Ising Model for the λ Transition and Phase Separation in He^3–He^4 Mixtures, *Phys. Rev. A* **4**:1071–1077 (1971).
3. T. D. Lee and C. N. Yang, Statistical Theory of Equations of State and Phase Transitions, II Lattice Gas and Ising Model, *Phys. Rev.* **87**:410–419 (1952).
4. R. B. Griffiths, C. A. Hurst, and S. Sherman, Concavity of Magnetization of an Ising Ferromagnet in a Positive External Field, *J. Math. Phys.* **11**:790–795 (1970).

5. B. Simon and R. B. Griffiths, The $(\varphi^4)_2$ Field Theory as a Classical Ising Model, *Comm. Math. Phys.* **33**:145–164 (1973).
6. C. M. Newman, Classifying General Ising Models, in *Les Méthodes Mathématiques de la Théorie Quantique des Champs*, F. Guerra, D. W. Robinson, and R. Stora, eds., C.N.R.S., Paris (1976), pp. 273–288.
7. C. M. Newman, Zeros of the Partition Function for Generalized Ising Systems, *Comm. Pure Appl. Math.* **27**:143–159 (1974).
8. C. M. Newman, Fourier Transforms with Only Real Zeros, *Proc. AMS*, to appear.
9. C. M. Newman, Inequalities for Ising Models and Field Theories which Obey the Lee–Yang Theorem, *Comm. Math. Phys.* **41**:1–9 (1975).
10. R. S. Ellis and C. M. Newman, Necessary and Sufficient Conditions for the GHS Inequality with Applications to Analysis and Probability, *Trans. AMS*, to appear.
11. R. S. Ellis, J. L. Monroe, and C. M. Newman, The GHS and Other Correlation Inequalities for a Class of Even Ferromagnets, *Comm. Math. Phys.* **46**:167–182 (1976).
12. J. L. Lebowitz, GHS and Other Inequalities, *Comm. Math. Phys.* **35**:87–92 (1974).
13. G. S. Sylvester, Inequalities for Continuous-Spin Ising Ferromagnets, *J. Stat. Phys.*, to appear.
14. R. S. Ellis and C. M. Newman, Quantum Mechanical Soft Springs and Reverse Correlation Inequalities, *J. Math. Phys.*, to appear.
15. C. M. Newman, Gaussian Correlation Inequalities for Ferromagnets, *Z. Wahrscheinlichkeitstheorie* **33**:75–93 (1975).
16. J. K. Percus, Correlation Inequalities for Ising Spin Lattices, *Comm. Math. Phys.* **40**:283–308 (1975).
17. G. S. Sylvester, Representations and Inequalities for Ising Model Ursell Functions, *Comm. Math. Phys.* **42**:209–220 (1975).
18. P. Cartier, Seminaire Bourbaki lecture, to appear.
19. R. B. Griffiths, Rigorous Results for Ising Ferromagnets of Arbitrary Spin, *J. Math. Phys.* **10**:1559–1565 (1969).
20. F. Dunlop and C. M. Newman, Multicomponent Field Theories and Classical Rotators, *Comm. Math. Phys.* **44**:223–235 (1975).
21. F. Dunlop, Correlation Inequalities for Multicomponent Rotators, IHES preprint (1975).
22. H. Kunz, Ch.-Ed. Pfister, and P. A. Vuillermot, Inequalities for Some Classical Spin Vector Models, ZIF Bielefeld preprint (1976).
23. J. Bricmont, Correlation Inequalities for Two Component Fields, Louvain preprint (1976).